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Chern-Simons Terms on Noncommutative Branes

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ABSTRACT

We write down couplings of the fields on a single BPS Dp-brane with noncommutative world-volume coordinates to the RR-forms in type II theories, in a manifestly background independent way. This generalises the usual Chern-Simons action for a commutative Dp-brane. We show that the noncommutative Chern-Simons terms can be mapped to Myers terms on a collection of infinitely many D-instantons. We also propose Chern-Simons couplings for unstable non-BPS branes, and show that condensation of noncommutative tachyons on these branes leads to the correct Myers terms on the decay products.

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Contents

1. Introduction and Review

Much insight has been gained into the dynamics of brane decay using noncommutativity. Turning on a constant NS-NS 2-form B-field along the world-volume of a D-brane, one finds that the world-volume action becomes a noncommutative field theory[1,2]. On an unstable, non-BPS D-brane, one gets a noncommutative field theory involving tachyonic scalars, which generically possesses noncommutative static soliton solutions[3] at large values of the noncommutativity parameter. These solutions can be used[4,5] to describe the decay of these branes. Subsequently it was understood that large noncommutativity is irrelevant to the problem, if one chooses the correct solution including gauge field excitations along with the tachyon[6,7]. A beautiful background-independent formulation of this, exhibiting the relation to matrix theory, was given by Seiberg recently[8].

Many of the above works deal with bosonic D-branes, which have a DBI action similar to that for the bosonic fields of D-branes in superstring theory. However, a unique property of branes in superstring theory is that they all have topological couplings of Chern-Simons type to the Ramond-Ramond closed-string backgrounds. In what follows, we will propose explicit expressions for the Chern-Simons couplings on BPS D-branes in the presence of noncommutativity.

We will also find analogous couplings on non-BPS branes, and show that these terms provide a useful testing ground for the conjectures involving brane decay via noncommutative solitons. One key property of noncommutative solitons is that they can produce N lower D-branes starting from a single higher brane. In recent times it has emerged that collections of N D-branes have extra commutator couplings in their world-volume theory, to all RR potentials, that do not appear for a single D-brane[9]. Hence one should expect to find such terms starting with a noncommutative D-brane. We will see that such terms indeed arise.

A (Euclidean) Dp-brane in bosonic string or superstring theory has a Dirac-Born-Infeld action for massless modes that can be written

$$S_{DBI} = T_p \int d^{p+1}x \, \sqrt{\det\left(g_{ij} + F_{ij} + B_{ij}\right)} \tag{1.1}$$

where $i = 1, \ldots, p+1$. g_{ij} and B_{ij} are pull-backs of the spacetime metric and NS-NS B-field onto the brane world-volume, $T_p = \frac{(2\pi)^{\frac{1-p}{2}}}{g_s}$, and we have set $2\pi\alpha' = 1$. We are working in static gauge and in the DBI limit, i.e. with constant fields.

The relevant results of ref.[2] can be summarised as follows. Let us assume for definiteness that we are dealing with a Euclidean Dp-brane with an even-dimensional world-volume (so p is odd) and a constant non-zero background B-field has been turned on over all $p+1$ directions. In the presence of this B-field, the dynamics of the world-volume fields of the brane is equivalently described by an action in terms of noncommutative variables given by

$$\hat{S}_{DBI} = \hat{T}_p \int d^{p+1}x \, \sqrt{\det\left(G_{ij} + \hat{F}_{ij} + \Phi_{ij}\right)} \tag{1.2}$$

where $\hat{T}_p = \frac{(2\pi)^{\frac{1-p}{2}}}{G_s}$, and products of fields appearing in this equation are understood to be $*$ products with parameter θ:

$$f(x) * g(x) \equiv e^{\frac{i}{2}\theta^{ij}\partial_i\partial'_j} f(x) g(x') \big|_{x=x'} \tag{1.3}$$

The new field strength $\hat{F}$ is given in terms of the commutative gauge field by the Seiberg-Witten transform. In the case of a single Dp-brane with constant $\hat{F}$ (rank one commutative gauge field F), $\hat{F}$ is related to F by

$$F = \hat{F}\frac{1}{1-\theta\hat{F}} \tag{1.4}$$

The parameters G_{ij}, Φ_{ij}, G_s and θ^{ij} are given in terms of the commutative variables g, B, g_s by

$$\begin{aligned} \frac{1}{G+\Phi} &= -\theta + \frac{1}{g+B} \\ G_s &= g_s\left(\frac{\det(G+\Phi)}{\det(g+B)}\right)^{\frac{1}{2}} \end{aligned} \tag{1.5}$$

Because these equations determine four parameters in terms of three physical (constant) backgrounds, there is a freedom in the description. This can be seen either as the freedom to choose the noncommutativity parameter θ at will (in which case G_{ij}, Φ_{ij}, G_s are determined), or the freedom to choose Φ_{ij}, in which case G_{ij}, θ^{ij}, G_s are determined).

Three choices of the description are particularly interesting. For $\Phi_{ij} = B_{ij}$, we have $\theta = 0, G_s = g_s$ and $G_{ij} = g_{ij}$. This is the commutative description. For $\Phi = 0$ we have some definite value of θ and the remaining parameters, such that all the noncommutativity is absorbed into $\hat{F}$ and the $*$ product. And for $\Phi_{ij} = -B_{ij}$, we have the special values:

$$\theta^{ij} = (B^{-1})^{ij}, \qquad G_{ij} = -B_{ik} g^{kl} B_{lj}, \qquad G_s = g_s \sqrt{\frac{\det B}{\det g}} \tag{1.6}$$

This last choice has the special feature of manifest background independence. This means the following: remaining within the description via $\Phi = -B$, we can vary B, keeping g, g_s and $F + B$ fixed. Then it was shown[2] that the DBI action is invariant under this change. Let us rewrite the DBI action in a way that makes this background-independence manifest.

Substituting $\Phi = -B = -\theta^{-1}$ into Eq.(1.2), we find that in this description:

$$\begin{aligned} \hat{S}_{DBI} &= \hat{T}_p \int d^{p+1}x \, \sqrt{\det\left(G_{ij} + \hat{F}_{ij} - \theta^{-1}_{ij}\right)} \\ &= T_p \sqrt{\det g} \sqrt{\det \theta} \int d^{p+1}x \, \sqrt{\det\left(- \theta^{-1}_{ik} g^{kl} \theta^{-1}_{lj} + \hat{F}_{ij} - \theta^{-1}_{ij}\right)} \\ &= T_p \int d^{p+1}x \, \frac{\mathrm{Pf}\, Q}{\mathrm{Pf}\, \theta} \sqrt{\det\left(g_{ij} + (Q^{-1})_{ij}\right)} \end{aligned} \tag{1.7}$$

where Pf denotes the Pfaffian of an antisymmetric matrix, and we have defined

$$Q^{ij} = \theta^{ij} - \theta^{ik} \hat{F}_{kl} \theta^{lj} \tag{1.8}$$

Note that this differs by a sign from the corresponding definition in Ref.[2].

It turns out that Q^{ij} is background-independent in the sense explained above[2]. This follows from Eqs.(1.4) and (1.6), which give:

$$F + B = \hat{F} \frac{1}{1 - \theta \hat{F}} + \frac{1}{\theta} = \frac{1}{Q} \tag{1.9}$$

The final step is to note[3] that the integral over noncommutative variables x^i satisfying

$$[x^i, x^j] = i\theta^{ij} \tag{1.10}$$

can be re-expressed in terms of a trace Tr over a Hilbert space of operators whose commutation relations are independent of θ. The transcription is:

$$\int d^{p+1}x \to \mathrm{Tr}\, (2\pi)^{\frac{p+1}{2}} \mathrm{Pf}\, \theta \tag{1.11}$$

from which we finally get:

$$\hat{S}_{DBI} = \frac{2\pi}{g_s} \mathrm{Tr}\,\mathrm{Pf}\,Q \sqrt{\det\left(g_{ij} + (Q^{-1})_{ij}\right)} \tag{1.12}$$

In this form, the background independence is manifest, from the background-independence of Q^{ij} and the fact that all the other quantities in this action are closed-string quantities, which are manifestly B-independent.

We can go one step further to obtain a useful insight into the meaning of this action. Since $\hat{F}$ and therefore Q is constant, we can try to find coordinates X^i which satisfy

$$[X^i, X^j] = iQ^{ij} \tag{1.13}$$

Then we can use Eq.(1.11) in the reverse direction, with the noncommutativity parameter being Q instead of θ:

$$\mathrm{Tr} \rightarrow \frac{1}{(2\pi)^{\frac{p+1}{2}}\,\mathrm{Pf}\,Q} \int d^{p+1}X \tag{1.14}$$

This enables us to write the action Eq.(1.12) schematically as:

$$\hat{S}_{DBI} = T_p \int d^{p+1}X \; \sqrt{\det\left(g_{ij} + F_{ij} + B_{ij}\right)} \tag{1.15}$$

where we have also used Eq.(1.9). This deceptively simple expression is just the *original* DBI action, in the original variables, but with all products replaced by $*$ products with noncommutativity parameter $Q = \frac{1}{F+B}$. The complication resides in the fact that it has to be interpreted as an action for the dynamical variable $\hat{A}$, which is defined in terms of $F + B$ via Eqs.(1.9) and (1.8).

Thus there is a "short route" from the commutative action Eq.(1.1) to the manifestly background-independent noncommutative action Eq.(1.15), bypassing the conversion to a formalism with Φ-dependence and the introduction of an "open-string" metric and string coupling. It consists of introducing noncommuting coordinates X and interpreting their noncommutativity parameter as the Q defined in Eq.(1.8). Alternatively, the prescription can be defined to lead to the last line of Eq.(1.7), namely insert a factor of $\frac{\mathrm{Pf}\,Q}{\mathrm{Pf}\,\theta}$ inside the integral, and replace all products by $*$ products with parameter θ^{ij}. In either case, one must replace $(F + B)_{ij}$ wherever it occurs by $(Q^{-1})_{ij}$.

It is also straightforward to understand the relation between the new X satisfying Eq.(1.13) and the old ones satisfying Eq.(1.10). This relation is

$$X^i = x^i + \theta^{ij}\hat{A}_j(x) \tag{1.16}$$

It was discovered in Refs.[10,11] and used recently in Ref.[8] to argue a relation between matrix theory and noncommutativity, exhibiting the role of matrix theory in ensuring background-independence. For our purposes it is not necessary to invoke matrix theory per se. The mere existence of the X^i variables gives us a recipe to find a noncommutative DBI action equivalent to the original commutative one, namely replacing x^i by X^i.

In the following section we will apply this recipe to finding the Chern-Simons terms for a noncommutative D-brane in type II superstring theory.

2. Chern-Simons Terms for a Noncommutative BPS D-brane

It is convenient to start with a single Euclidean BPS D9-brane of type IIB theory. We follow the normalization of Ramond-Ramond forms as in Ref.[9]. Let μ_p be the RR charge (which equals the tension T_p in these conventions) of a BPS Dp-brane. With these conventions, the Chern-Simons terms on a D9-brane are:

$$S_{CS} = \mu_9 \int \sum_n C^{(n)}\, e^{B+F} \tag{2.1}$$

where $C^{(n)}$ denotes the n-form RR potential. As is well-known, the above expression involves the following prescription: the exponential is to be expanded in a formal power series of wedge products, and each term is then wedged with the appropriate RR form so that the total form dimension is 10.

In the presence of non-zero constant NS-NS B-field background, we have seen that the DBI action has a manifestly background-independent description in which the world-volume becomes a noncommutative space with noncommutativity parameter $Q = \frac{1}{F+B}$. Following the procedure outlined in the previous section, we now write down the Chern-Simons terms on the noncommutative D9-brane. Thus, in the previous equation we simply replace $(F+B)_{ij}$, wherever it occurs, by $(Q^{-1})_{ij}$, and insert a factor $\frac{\mathrm{Pf}\,Q}{\mathrm{Pf}\,\theta}$ under the integral sign. Then the noncommutative Chern-Simons term is:

$$\hat{S}_{CS} = \mu_9 \int_x \frac{\mathrm{Pf}\,Q}{\mathrm{Pf}\,\theta} \sum_n C^{(n)}\, e^{\,Q^{-1}} \tag{2.2}$$

where the underlying coordinates are the x^i and all the products are understood to be $*$ products defined in terms of θ. Here Q^{-1} is understood to be the 2-form $\frac{1}{2}(Q^{-1})_{ij}\, dx^i \wedge dx^j$.

Just as was done for the DBI action, this can alternatively be expressed as the Hilbert-space trace:

$$\hat{S}_{CS} = \frac{2\pi}{g_s} \mathrm{Tr}\,\mathrm{Pf}\, Q \ \sum_n C^{(n)}\, e^{\,Q^{-1}} \tag{2.3}$$

This can also be re-expressed schematically in terms of "original" variables as:

$$\hat{S}_{CS} = \mu_9 \int_X \sum_n C^{(n)} e^{F+B} \tag{2.4}$$

where this time the integral is over coordinates X^i with noncommutativity parameter Q.

The modification of the Chern-Simons terms on a D9-brane due to noncommutativity can be understood physically as follows. Suppose that, to start with, we turn on a constant B field on the D9-brane only along the directions (x^9, x^{10}). In noncompact space, this effectively induces infinitely many D7-branes on the world-volume of the D9-brane. But we know that such a collection of D7-branes must couple to the RR 10-form potential via Myers terms[9], see also Refs.[12,13]. Hence the expression we have proposed in Eq.(2.3) must contain these terms. Indeed, since we have taken maximal noncommutativity there, our situation is more like that of infinitely many D-instantons, which explains why the integral has been completely replaced by a trace over an infinite dimensional Hilbert space. This action then should contain Myers terms describing the coupling of the D-instantons to all p-form potentials (even p) in type IIB theory.

To see that this is so, let us first examine the term containing the 10-form $C^{(10)}$. The Myers term proportional to this form, on a collection of N D-instantons, looks like:

$$\frac{2\pi}{g_s} \mathrm{tr}\, e^{i(\mathrm{i}_\Phi \mathrm{i}_\Phi)} C^{(10)} \tag{2.5}$$

where i_Φ is interpreted to mean the inner product of the $N \times N$ matrix-valued scalar field Φ^i transverse to the brane, with a lower index on the RR potential. Expanding the exponential, we find the relevant Myers term is:

$$\begin{aligned} \frac{2\pi}{g_s} \mathrm{tr}\, e^{i(\mathrm{i}_\Phi \mathrm{i}_\Phi)} C^{(10)} &= \frac{2\pi}{g_s} \mathrm{tr}\, \frac{i^5}{5!}\, \Phi^{i_{10}} \Phi^{i_9} \ldots \Phi^{i_2} \Phi^{i_1}\, C^{(10)}_{i_1 i_2 \cdots i_9 i_{10}} \\ &= -\frac{2\pi}{g_s} \mathrm{tr}\, \frac{1}{5! 2^5} \left(i[\Phi^{i_1}, \Phi^{i_2}] \right) \ldots \left(i[\Phi^{i_9}, \Phi^{i_{10}}] \right) C^{(10)}_{i_1 i_2 \cdots i_9 i_{10}} \end{aligned} \tag{2.6}$$

To compare, let us extract the 10-form term from Eq.(2.3):

$$\begin{aligned}\frac{2\pi}{g_s}\mathrm{Tr}\,(\mathrm{Pf}\,Q)\,C^{(10)} &= \frac{2\pi}{g_s}\mathrm{Tr}\,\frac{1}{5!2^5}\,\epsilon_{i_1 i_2\cdots i_9 i_{10}}Q^{i_1 i_2}\ldots Q^{i_9 i_{10}}\frac{1}{10!}\epsilon^{j_1 j_2\cdots j_9 j_{10}}\,C^{(10)}_{j_1 j_2\cdots j_9 j_{10}} \\ &= \frac{2\pi}{g_s}\mathrm{Tr}\,\frac{1}{5!2^5}\,Q^{i_1 i_2}\ldots Q^{i_9 i_{10}}\,C^{(10)}_{i_1 i_2\cdots i_9 i_{10}}\end{aligned} \tag{2.7}$$

Now in noncommutative theory, as we have seen, the identification with matrix-valued transverse coordinates comes about as $Q^{ij} = -i[X^i, X^j]$. Identifying the X^i with Φ^i, we see that the two expressions above agree.

Note that the above manipulations are covariant without specifying a spacetime metric: Q naturally has upper indices, while $C^{(10)}$ is a 10-form and has lower indices. The ϵ symbols above have constant components ± 1, so they are tensor densities (the upper- and lower-indexed ones having equal and opposite weight). Hence the action is always a true scalar.

One can actually display the equivalence of the noncommutative action Eq.(2.3) to the Myers terms on a D-instanton in complete generality. Like the commutative Chern-Simons terms in Eq.(2.1), the noncommutative version Eq.(2.3) also involves a prescription whereby the exponential of the 2-form Q^{-1} is expanded and its wedge products taken with the appropriate RR form to make 10-forms. Since the integral has now been converted to a trace, these 10-forms are contracted with the totally antisymmetric ϵ-tensor in 10 dimensions (whose components are ± 1) to make 0-forms. By manipulations similar to those above, it is then easy to derive the following identity:

$$\mathrm{Tr}\,\mathrm{Pf}\,Q\,\sum_n C^{(n)} e^{\,Q^{-1}} = \mathrm{Tr}\,e^{-\frac{1}{2}\mathrm{i}_Q}\sum_n C^{(n)} \tag{2.8}$$

where i_Q acting on a 2-form ω_{ij} is defined as $Q^{ji}\omega_{ij}$, and the prescription on the RHS is as follows: the exponential is expanded out to such an order that the n-form on which it acts is reduced by contractions to a scalar. This is precisely Myers' prescription applied to D-instantons! Thus we see that noncommutativity can be used to derive the Myers-Chern-Simons terms, with the "dotting" prescription of Myers being dual to the wedge prescription arising in conventional Chern-Simons terms on branes. This in particular demonstrates that the expression Eq.(2.3) is nonsingular despite the appearance of Q^{-1}.

Let us next consider lower BPS D-branes, for example a (Euclidean) D7-brane in type IIB. The noncommutativity parameter θ^{ij} is now chosen to be maximal with respect to the eight Euclidean directions $x^1, x^2, \ldots, x^8$. Therefore this D7-brane is T-dual to a D9-brane with noncommutativity only along the first eight directions. Note that on this D7-brane we now have transverse coordinates Φ^9, Φ^{10} that are functions of the noncommuting brane world-volume coordinates $x^i, i = 1, \ldots, 8$. It follows that Φ^9 and Φ^{10} do not commute, or alternatively that they are multiplied using the $*$ product. This suggests that in addition to the Q-dependent modification to the Chern-Simons term as in Eq.(2.2), we also have Myers-type terms involving Φ^9, Φ^{10}.

Thus we claim that the Chern-Simons term on the D7-brane is:

$$\hat{S}_{CS} = \mu_7 \int_x \frac{\mathrm{Pf}\, Q}{\mathrm{Pf}\, \theta}\, P \left[e^{i(\mathrm{i}_\Phi * \mathrm{i}_\Phi)} \sum_n C^{(n)} \right] e^{\,Q^{-1}} \tag{2.9}$$

where P represents the pull-back of the transverse brane coordinates to the world-volume (this pullback was not required while studying the 9-brane, since there were no directions transverse to the brane in that case). Note that here, Q is an 8×8 rather than 10×10 matrix.

This expression is basically equivalent to Eq.(2.2). To see this, notice that the exponential term $e^{i(\mathrm{i}_\Phi * \mathrm{i}_\Phi)}$ is equivalent to $1 + i(\mathrm{i}_\Phi * \mathrm{i}_\Phi)$ since higher powers cannot contribute by antisymmetry of the object that they contract. If we convert the 8-dimensional integral above into a Hilbert-space trace, and define $Q^{9,10} = -i[\Phi^9, \Phi^{10}]$, then it is not hard to see that Eq.(2.9) becomes equivalent to Eq.(2.2). For example, in Eq.(2.9), the term $(Q^{-1})_{9,10}$ is missing, but the term in Eq.(2.2) where it would contribute by cancelling a factor of $Q^{9,10}$ from the Pfaffian, arises instead from the 1 in $1 + i(\mathrm{i}_\Phi * \mathrm{i}_\Phi)$.

Thus we see that Chern-Simons terms on BPS D-branes in the presence of noncommutativity are summarised by Eq.(2.2), though for different p they are more appropriately written in terms of a hybrid of $\mathrm{Pf}\, Q$ for the directions within the brane, and exponential terms of the Myers type for the directions transverse to the brane, as in Eq.(2.9) for the case of the D7-brane.

3. Chern-Simons Terms for a Noncommutative Non-BPS D-brane

We now turn to the discussion of unstable, non-BPS D-branes and determine the Chern-Simons action on their world-volumes in the presence of noncommutativity. Type II superstring theories have unstable Dp-branes, where p is odd in type IIA and even in type IIB. Like the BPS branes, these branes too have Chern-Simons terms on their world-volumes[14,15,16,17,18] which play an important role in arguing that their decay products (in the commutative case) carry the right RR charges. However, these Chern-Simons terms now depend on the tachyon field as well.

For a single unstable commutative p-brane, the CS term has been argued to be:

$$S_{CS} = \frac{\mu_{p-2}}{2T_{min}} \int dT\, C^{(n)}\, e^{F+B} \tag{3.1}$$

where T_{min} is the value of the tachyon at the minimum of the potential. Note that dT is a 1-form, and the prescription for the Chern-Simons term involves expanding the exponential so that, after wedging with an appropriate RR potential *and* the tachyon factor dT, we get a 10-form.

For the noncommutative case, let us work with a Euclidean D9-brane in type IIA with noncommutativity over all the 10 directions. Following the procedure described in the previous sections, we can try to write down the Chern-Simons action. However, there is an important subtlety to be taken care of. The exterior derivative dT of the tachyon in the above equation must be promoted to a covariant derivative in the noncommutative case. The covariant derivative in our notation is $D_i = \theta^{-1}_{ij} X^j$ with X^i defined in Eq.(1.16). The problem is that since X^i is background-independent, D_i depends explicitly on θ. We need to find a 1-form in place of dT that is linear in $[X^i, T]$ and also background-independent. The unique candidate seems to be

$$\mathcal{D}_i T = -i\,(Q^{-1})_{ij}[X^j, T] \tag{3.2}$$

where Q has been defined in Eq.(1.8). One piece of encouragement for this replacement, besides background-independence, comes from the fact that at $\hat{F} = 0$, Q is the same as θ, so our covariant derivative reduces to the standard one in that case. We will see another justification for it in the following section.

Accordingly, we propose the following Chern-Simons action for an unstable D9-brane:

$$\hat{S}_{CS} = \frac{\mu_8}{2T_{min}} \int_x \frac{\mathrm{Pf}\, Q}{\mathrm{Pf}\, \theta}\, \mathcal{D}T\, C^{(n)}\, e^{\,Q^{-1}} \tag{3.3}$$

with $\mathcal{D}T$ as defined above. We will see in the following section that this modification is crucial in reproducing expected results after brane decay via noncommutative solitons.

Note that even the commutative Chern-Simons term of Eq.(3.1) could have higher-order corrections containing powers of T, as noted in Refs.[19,18]. In this case, the noncommutative version will also generalise in an obvious way.

4. Noncommutative Solitons, Brane Decay and Myers Terms

Now we can use the known classical solutions for noncommutative tachyons representing lower dimensional unstable D-branes, and study the Chern-Simons action expanded around these solutions.

Let us first review the relevant information in Ref.[7], which extended the observations in Refs.[4,5], about noncommutative soliton solutions on unstable D-branes. However, unlike these references, we work in the description $\Phi_{ij} = -B_{ij}$ following Seiberg[8]. In this description the action is manifestly independent of both θ and B. Written as a trace over a Hilbert space, the DBI action for an unstable D9-brane is:

$$\hat{S}_{DBI} = \frac{2\pi}{g_s} \mathrm{Tr} \left[V(T)\sqrt{\det\left(\delta_i^j - i g_{ik}[X^k, X^j]\right)} - f(T)\, g_{kl}\, [X^k, T][X^l, T] + \cdots \right] \tag{4.1}$$

where $V(T)$ is the tachyon potential, and all products are understood to be $*$ products. $f(T)$ is some function whose form we will not need, although from Refs.[20,21] it follows that it is proportional to $V(T)$.

The above action can be obtained following the "long route" described in Section 1. However, it can also be obtained via the short route if we assume the formula in Eq.(3.2). For this, we start with the commutative action in the form[20,21]:

$$S_{DBI} = T_p \int d^{p+1}x\ \sqrt{\det\left(g_{ij} + F_{ij} + B_{ij} + \partial_i T \partial_j T\right)} \tag{4.2}$$

Using the prescription in Section 1 along with the one in Eq.(3.2), the noncommutative action written as a trace is:

$$\hat{S}_{DBI} = \frac{2\pi}{g_s} \mathrm{Tr}\, \mathrm{Pf}\, Q \sqrt{\det\left(g_{ij} + (Q^{-1})_{ij} + \mathcal{D}_i T \mathcal{D}_j T^\dagger\right)} \tag{4.3}$$

Taking the Pfaffian factor inside the square root and expanding, we easily recover Eq.(4.1), with $f(T) = V(T)$[1]. This provides additional confirmation of Eq.(3.2).

Now one can extremise this action and find the classical equations of motion for both the operators T and X^i. For the equation of motion of the tachyon field we get:

$$V'(T)\sqrt{\det M} - f'(T)\, g_{kl}\, [X^k, T][X^l, T] + 2 g_{kl}\, [X^k, [X^l, T]\, f(T)] = 0 \tag{4.4}$$

[1] It is important to note that the expansion of the DBI action is being carried out for $|g|, |\partial T|^2 \ll |B + F|$, while in the commutative theory the DBI action is expanded in the opposite regime, where $|\partial T|^2, |B + F| \ll |g|$.

and for the X^i we get:

$$2g_{ij}\left[T,\left[T,X^j\right]f(T)\right]+ig_{kl}\left[X^l,V(T)\sqrt{detM}(M^{-1})_i^k\right]=0 \tag{4.5}$$

where we have defined:

$$(M)_i^j=\delta_i^j-ig_{ik}\left[X^k,X^j\right] \tag{4.6}$$

The equations of motion above actually suffer from an ordering problem, as does the DBI action itself. However, as was essentially noted in Ref.[7], the classical solutions representing noncommutative solitons turn out to be independent of any rearragements of terms one might make above (so long as one does not, of course, open out commutators).

In these variables we can find solutions for these equations that correspond to the "nothing state" and to various lower dimensional branes. As was pointed out by various authors[7,8], each such state has in general a multitude of solutions. Recently Sen[22] has argued that this apparent degeneracy of solutions arises because the variables in which the DBI action is written are not the correct variables at the end-point of tachyon condensation. One has to use some combinations of these variables to get rid of the unwanted degeracy of solutions. For our purpose, however, it is enough to carry out the analysis using any one of the many physically equivalent solutions.

Hence we take the following solution as the "nothing state":

$$\begin{aligned}T_{\rm cl}&=T_{min}1\\X^i_{\rm cl}&=0\qquad\text{for}\qquad i=1,2,\ldots,10\end{aligned} \tag{4.7}$$

where 1 represents the identity operator. For a codimension-two soliton representing a $D7$ brane (say along the $x^1,x^2,\ldots,x^8$ directions), we choose the following solution:

$$\begin{aligned}T_{\rm cl}&=T_{max}P_N+T_{min}(1-P_N)\\X^i_{\rm cl}&=P_Nx^i\quad\text{for}\quad i=1,2,\ldots,8\\X^i_{\rm cl}&=0\quad\text{for}\quad i=9,10\end{aligned} \tag{4.8}$$

where T_{max} and T_{min} are the values of the tachyon field at the extrema of $V(T)$, and P_N is the level-N projection operator in the harmonic oscillator Hilbert space made out of the noncommutative directions x^8,x^9.

It is easy to see that for the "nothing state" solution Eq.(4.7), the action vanishes identically. For the codimension-two soliton, using:

$$V\Big(T_{max}P_N + T_{min}(1-P_N)\Big) = V(T_{max})\,P_N \tag{4.9}$$

we find that the action is given by:

$$\begin{aligned}\hat{S} &= \frac{2\pi}{g_s}\mathrm{Tr}\,V(T_{max})P_N\sqrt{\det\left(\delta_i{}^j + P_N g_{ik}\theta^{kj}\right)}\\ &= \frac{2\pi}{g_s}\,N\sqrt{\det\left(\delta_i{}^j + g_{ik}\theta^{kj}\right)}\end{aligned} \tag{4.10}$$

This expression is identified with the action for N unstable $D7$ branes with all fluctuations set to zero (in particular, this means that $T = T_{max}$ and $\hat{A} = 0$).

The apparent θ-dependence of this result is understood as follows. We have obtained the action for unstable D7-branes in the description with $\Phi = -\theta^{-1}$, and at $\hat{F} = 0$. For general Φ the answer should be proportional to $\sqrt{\det(G+\Phi)}$. Hence in the background-independent description, with $\Phi = -\theta^{-1}$, it should be proportional to $\sqrt{\det(G-\theta^{-1})}$, which is the case. Another way to put it is that the θ in Eq.(4.10) is really Q, as defined in Eq.(1.8), and evaluated at $\hat{F} = 0$.

We could in fact have chosen the classical solution for X^i to be given by $X^i_{\rm cl} = P_N\overline{x}^i$ for arbitrary $\overline{x}^i$ satisfying $[\overline{x}^i, \overline{x}^j] = i\overline{Q}^{ij}$, this would have reproduced D7-branes in the state with $Q^{ij} = \overline{Q}^{ij}$. But the state with $\hat{A} = 0$ (hence $\overline{Q}^{ij} = \theta^{ij}$) is special in that it describes the final D7-branes in their undecayed state.

We can now insert these solutions into the Chern-Simons term of Eq.(3.3) and find the Chern-Simons action for the fluctuations of these solitons. Here we will just demonstrate how to get Chern-Simons terms for an unstable D7 brane of type IIA. For this, let us first consider a specific term from Eq.(3.3) for an unstable D9 brane:

$$\hat{S}_{CS} = \frac{\mu_8}{2T_{min}}\int_x \frac{\mathrm{Pf}\,Q}{\mathrm{Pf}\,\theta}\,(-i)(Q^{-1})_{ij}[X^j, T]\,C^{(9)} \tag{4.11}$$

Now we condense the noncommutative tachyon as a codimension two soliton given in Eq.(4.8) to obtain N D7-branes along $x^1, x^2, ..., x^8$ directions. We substitute

$$T = T_{\rm cl} + \delta T, \qquad X^i = X^i_{\rm cl} + \delta X^i \tag{4.12}$$

with the classical solutions defined in Eq.(4.8), into Eq.(4.11). In (4.12) we take the fluctuations δT and δX^i to be independent of the x^9, x^{10} directions. Then this CS action will have the form of an action on a set of D7-branes. Indeed, we saw in Eq.(4.10) that expanding the DBI action brings out a projection operator P_N for the Hilbert space along the noncommutative directions $9, 10$, so that the remaining $(\infty - N)$ modes become non-propagating. The same projection arises in the CS action because of the P_N in the classical solution for the X^i.

Notice that with this classical solution we have $[X^i_{\rm cl}, T_{\rm cl}] = 0$, so the Chern-Simons action of the classical solution vanishes identically. The decay product therefore has no "winding brane charge", contrary to the claim in Refs.[4,23]. This is particularly reassuring in view of a recent argument[24] that some of the configurations that were thought to carry such charge are pure gauge under a certain discrete symmetry.

Now let us consider the case in which the 9-form $C^{(9)}_{i_1 i_2 .. i_9}$ has its indices along the directions $2, 3, ..., 10$. In this case the index i in the 1-form $(Q^{-1})_{ij}[X^j, T]$ in Eq.(4.11) will have to be along the direction 1. In a coordinate basis in which the matrix Q^{ij} is in the canonical form, it follows that the index j must be 2. Then we end up with the following action for the fluctuations δT and δX^i:

$$\hat{S}^{UD7}_{CS} = \frac{\mu_6}{2T_{min}} {\rm tr}_N \int_x (-i)\,[\delta X^9, \delta X^{10}]\,(-i)(Q^{-1}_{\rm cl})_{12}\,[X^2_{\rm cl}, \delta T]\,C^{(9)}_{23\ldots 10} \tag{4.13}$$

where $Q^{ij}_{\rm cl} = \theta^{ij}$. Now in the above expression, X^9, X^{10} are $N \times N$ matrices (because of the P_N projection) and the Hilbert-space trace over the directions of the noncommutative soliton becomes a trace over these matrices. Hence, what we have found here is a Myers term on N unstable D7-branes.

Actually, the commutator $[\delta X^9, \delta X^{10}]$ that enters above does not vanish even for $N = 1$. This is because the X^i's appearing in it are not only $N \times N$ matrices, but also functions of the remaining 8 coordinates which are multiplied using the $*$ product. This is due to our choice of noncommutativity over all the directions, and not just over the two directions along the noncommutative soliton solution and transverse to the final D7-brane.

Next we may consider the term where the indices on $C^{(9)}$ are $1, 2, \ldots, 8, 10$. In this case, i in Eq.(4.11) must be 9 and therefore j is 10. The resulting term is:

$$\begin{aligned} \hat{S}^{UD7}_{CS} &= \frac{\mu_6}{2T_{min}} {\rm tr}_N \int_x (-i)\,[\delta X^9, \delta X^{10}]\,(-i)(\delta Q^{-1})_{9,10}\,[\delta X^{10}, \delta T]\,C^{(9)}_{12\ldots 8,10} \\ &= \frac{\mu_6}{2T_{min}} {\rm tr}_N \int_x (-i)[\delta X^{10}, \delta T]\,C^{(9)}_{12\ldots 8,10} \end{aligned} \tag{4.14}$$

where $\delta Q^{9,10}$ is shorthand for $-i[\delta X^9, \delta X^{10}]$. This is a new kind of Myers term on N unstable branes, that has no analogue for BPS branes. It was discovered very recently in Ref.[18].

We see that the Chern-Simons terms on a noncommutative unstable D-brane, Eq.(4.11), beautifully reproduce the structure of extra commutator terms that are expected to be present for an assembly of N unstable D-branes, lending further support to the idea that noncommutative tachyons really do reproduce N unstable D-branes of codimension 2 and that the $U(N)$ of these D-branes is naturally embedded in the $U(\infty)$ associated to noncommutativity.

5. Discussion and Conclusions

We have found Chern-Simons actions for D-branes of type IIB superstring theory, with noncommutativity (a B-field) along their world-volume. These actions are manifestly background-independent, and also manifestly gauge-invariant as they are expressed in terms of traces over a Hilbert space with $U(\infty)$ symmetry. The noncommutative expressions are elegant and turn out to contain all information about Myers terms on multiple branes.

In our work we focussed on Euclidean branes with maximal noncommutativity, hence branes of even world-volume dimension. This means that we studied BPS branes in type IIB, and unstable branes in type IIA. The other cases: BPS branes in type IIA and unstable branes in type IIB, can be studied by remaining in Minkoswki signature and turning on a B-field over all the spatial directions. In this case, the noncommutative actions will resemble actions for D0-branes rather than D-instantons.

The noncommutative Chern-Simons terms found above are valid, as for the Myers terms, in the static gauge for a D-brane with very slowly varying fields. A covariant generalisation of this should be expected to exist and could perhaps be found along the lines of Ref.[25]. Extension to higher-derivative terms in the field strength is also an interesting open question, for example one could try to generalise the results of Ref.[26] to the noncommutative case.

It would also be interesting to check how the noncommutative CS terms transform under T-duality — they should form a consistent collection, as for the case of the DBI action[20,21] and Chern-Simons action[9,18] on single or multiple branes. One can also hope to check that these terms, on unstable branes, give correct results when evaluated on codimension-one noncommutative solitons[27]. And, with our methods it should be straightforward to write down the Chern-Simons terms on a noncommutative brane-antibrane pair, generalising the result of Ref.[17], and study noncommutative solitons on these pairs[5,23,28].

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